

# Bias-Limited Extraction of Cosmological Parameters

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**Abstract.** It is known that modeling uncertainties and astrophysical foregrounds can potentially introduce appreciable bias in the deduced values of cosmological parameters. While it is commonly assumed that these uncertainties will be accounted for to a sufficient level of precision, the level of bias has not been properly quantified in most cases of interest. We show that the requirement that the bias in derived values of cosmological parameters does not surpass nominal statistical error, translates into a maximal level of *overall* error  $O(N^{-\frac{1}{2}})$  on  $|\Delta P(k)|/P(k)$  and  $|\Delta C_l|/C_l$ , where  $P(k)$ ,  $C_l$ , and  $N$  are the matter power spectrum, angular power spectrum, and number of (independent Fourier) modes at a given scale  $l$  or  $k$  probed by the cosmological survey, respectively. This required level has important consequences on the precision with which cosmological parameters are hoped to be determined by future surveys: In virtually all ongoing and near future surveys  $N$  typically falls in the range  $10^6 - 10^9$ , implying that the required overall theoretical modeling and numerical precision is already very high. Future redshifted-21-cm observations, projected to sample  $\sim 10^{14}$  modes, will require knowledge of the matter power spectrum to a fantastic  $10^{-7}$  precision level. We conclude that realizing the expected potential of future cosmological surveys, which aim at detecting  $10^6 - 10^{14}$  modes, sets the formidable challenge of reducing the overall level of uncertainty to  $10^{-3} - 10^{-7}$ .

**Keywords:** cosmological parameters from CMBR

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## 1 Introduction

The last two decades marked the beginning of the ‘precision cosmology’ era when many cosmological surveys are conducted in order to determine about a dozen cosmological parameters, and test non-standard cosmological models. These cosmological surveys are motivated by detailed theoretical predictions, elaborately designed satellite and stratospheric telescope systems, and computationally challenging data analysis and parameter inference procedures.

It is often stated that the joint effort of a variety of cosmological probes will enable breaking some of the vexing degeneracies in cosmological parameter space, but perhaps more importantly, it is hoped that cross-correlating the results obtained with a battery of cosmological probes could be used to mitigate inconsistencies due to the different systematics affecting each probe.

In this work we generalize basic arguments, first discussed by Seljak et al. [1] in the context of assessing the degree of agreement between different Boltzmann codes employed to calculate power spectra of the cosmic microwave background (CMB). We argue that as cosmological surveys become more advanced by virtue of higher spectro-spatial resolution and sensitivity, larger volume coverage, and lower instrumental noise, the need for a better understanding of the theory, higher numerical precision, a model-independent description of nonlinear effects, astrophysical foregrounds, and instrumental systematics, are all essential for achieving the full benefit of these advanced cosmological probes. The *quantitative* implications and ensuing ramifications of this (quite intuitive) statement are of primary interest for this work.

The challenge stems from the fact that the statistical error in inferred cosmological parameters decreases as  $\sim N^{-1/2}$ , where  $N$  is the number of independent Fourier modes that can be probed by a given experiment. In contrast, the bias (induced by e.g. inaccurate theory or model, systematics, and foreground removal) does not decrease with the number of modes. The problem then arises with those advanced probes that target a large number

of modes where the tension between the two trends becomes sufficiently large to result in an unavoidable bias in the inferred cosmological parameters.

Current cosmological probes access  $O(10^6) - O(10^9)$  modes, whereas next generation experiments aim at probing over  $10^{14}$  modes, most notably redshifted-21-cm observations, which allows 1-2 orders of magnitude tighter constraints to be placed on a few cosmological parameters (e.g. [2-5]). It is clear that either an unimaginable precision in theoretical modeling has to be achieved, or the number of modes used for parameter estimation must be cut at lower angular (2D) or spatial (3D) resolution than previously projected, thereby significantly degrading the scientific yield of these probes with respect to standard estimates.

The outline of the paper is as follows. In section 2 we provide a general discussion of biased parameter inference in the framework of Fisher matrix formalism, followed by applications to angular and matter power spectra. In section 3 we discuss a few specific examples of systematics and modeling uncertainties and highlight their relevance to biased parameter inference. Our conclusions are summarized in section 4.

## 2 Statistical Error and Bias

Analyses of cosmological surveys, such as galaxy correlations, galaxy shear, CMB, and supernovae (SNe) are commonly based on maximum likelihood methods to obtain the best fit cosmological model to the data. To assess bias in cosmological parameter extraction, we assume the likelihood function is gaussian in the power spectra. Specifically, this function is gaussian in the angular CMB power spectra, in the matter power spectrum of large scale structure (LSS) proxies (e.g. galaxy clustering, galaxy lensing, BAO,  $\text{Ly}\alpha$ , 21cm), and in the luminosity distance of SNe. While this function is poissonian in number counts of galaxy clusters, the numbers are sufficiently large to warrant a gaussian approximation.

In the framework of standard cosmology, it is clear that its inherent degree of symmetry (either the global 2D sky isotropy or 3D spatial homogeneity of the universe) implies that the universe is densely sampled on small angular (2D) and physical scales (3D), and thereby limited only by the fundamental minimum scale that could be accessed by a specific statistical probe with a given experiment (i.e. by the multipole number  $l$  and wavenumber  $k$  in the 2D and 3D cases, respectively, rather than the 2D and 3D wave-vectors  $\mathbf{l}$  and  $\mathbf{k}$ ). In the ideal case of no instrumental noise this implies that the only fundamental limit on precision is the so-called ‘cosmic-variance-limit’ (CVL), i.e. the fact that we observe only a single realization of the universe out of an infinitely large number of possible universes characterized by the same model (with all models having the exact same variance, i.e. power spectrum, but different one-point functions).

We begin quantitative assessment of statistical error and bias by a brief derivation of the well-known expressions for these measures within the Fisher matrix formalism. We then apply this formalism specifically to the angular and matter power spectra.

Without limiting the generality of the discussion we assume the likelihood function is gaussian in the data, at least in the large numbers limit. For a quantity  $d_n$  for which there are  $n_{max}$  data points  $\hat{d}_n$ , where  $n = 1, \dots, n_{max}$  stands for the multipole number  $l$  in the case of CMB and galaxy lensing,  $k$  and  $z$  in the case of large scale proxies, and redshift bin in the case of SNe and number counts, the likelihood function can then be written as

$$\mathcal{L} = \exp \left[ - \sum_{n=1}^{n_{max}} \frac{(d_n - \hat{d}_n)^2}{2(\delta d_n)^2} \right] \quad (2.1)$$

where  $\delta d_n$  is the statistical uncertainty on  $\hat{d}_n$  due to, e.g., observational, cosmic variance, and shot noise (in the case of galaxy lensing). Let us assume that there are  $M$  model parameters,  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_M)$ . Taylor expansion of  $d_n$  around the best fit model,  $\boldsymbol{\lambda}_0$ , keeping terms to first order, we obtain

$$d_n(\boldsymbol{\lambda}) \approx d_n(\boldsymbol{\lambda}_0) + \frac{d(d_n)}{d\boldsymbol{\lambda}} \cdot (\boldsymbol{\lambda} - \boldsymbol{\lambda}_0), \quad (2.2)$$

where we use boldface letters for either vectors or 2D matrices. The likelihood function, Eq.(2.1), is then,

$$\mathcal{L} \approx \exp \left[ -\frac{1}{2} \sum_{n=1}^{n_{max}} (\boldsymbol{\lambda} - \boldsymbol{\lambda}_0) \cdot \mathbf{F}_n \cdot (\boldsymbol{\lambda} - \boldsymbol{\lambda}_0) \right] \quad (2.3)$$

where for a given  $n$  we have defined

$$(\mathbf{F}_n)_{ij} \equiv \frac{1}{(\delta d_n)^2} \frac{d(d_n)}{d\lambda_i} \frac{d(d_n)}{d\lambda_j} \quad (2.4)$$

and  $(F)_{ij} = \sum_n (\mathbf{F}_n)_{ij}$  is the  $ij$  element of the familiar Fisher matrix. The statistical error on the parameter  $\lambda_i$  is

$$\sigma_{\lambda_i} = \sqrt{((\mathbf{F})^{-1})_{ii}}. \quad (2.5)$$

Assume that we have a modeling error, or numerical imprecision, and that our theoretical model shifts by  $\Delta d_n$ , i.e.  $d_n \rightarrow d_n + \Delta d_n$ , that is either unknown or otherwise cannot be quantified and accounted for. In the following we consider a single data point for notational simplicity, for which the likelihood function is

$$\mathcal{L} \rightarrow \mathcal{L}' = \exp \left[ -\frac{[\Delta d + \frac{d(d)}{d\boldsymbol{\lambda}} \cdot (\boldsymbol{\lambda} - \boldsymbol{\lambda}_0)]^2}{2(\delta d)^2} \right]. \quad (2.6)$$

Solving for the peak of the likelihood function, we obtain the shift, i.e. bias, in the best-fit parameters

$$\delta \boldsymbol{\lambda} \equiv \boldsymbol{\lambda} - \boldsymbol{\lambda}_0 = \frac{\Delta d}{(\delta d)^2} \mathbf{F}^{-1} \cdot \frac{d(d)}{d\boldsymbol{\lambda}}. \quad (2.7)$$

Now, assuming the bias and statistical error are uncorrelated we can add the nominal statistical error (Eq. 2.5) and the bias (Eq. 2.7) in quadrature

$$\sigma_{\lambda_i}^2 \rightarrow \sigma_{\lambda_i}^2 + (\delta \lambda_i)^2. \quad (2.8)$$

Comparing Eqs.(2.5) & (2.7) we see that in order for the model bias  $\Delta d$  to generate a significant parameter bias it has to be at the level of (or larger than) the variation of the theoretical  $d$  over the ‘allowed’ range of parameter values  $\Delta d \gtrsim \frac{d(d)}{d\boldsymbol{\lambda}} \sigma_{\boldsymbol{\lambda}}$ . In other words, while statistical error drops with increasing number of modes, any systematic bias in the theoretical model does not; therefore, for a given theoretical or modeling accuracy, there is a maximal mode beyond which the bias exceeds the statistical error. This implies that even if  $\Delta d$  is smaller than  $\hat{d}$  by a very large factor it can still result in a much more significant bias in parameter inference if the sensitivity of  $d$  to small variations in the cosmological model is large. In that case the bias in the data,  $\Delta d$ , can mimic parameter shift, resulting in a biased parameter inference.

## 2.1 CMB and Matter Power Spectra

To illustrate our basic argument in the context of the CMB it is sufficient to consider the temperature-only dataset. In the absence of instrumental or any other noise source we have

$$\delta C_l = \sqrt{\frac{1}{(2l+1)f_{sky}}} C_l, \quad (2.9)$$

which is the cosmic variance. Note that as the sky fraction,  $f_{sky}$ , is smaller, sample variance,  $\delta C_l$ , increases. In the presence of instrumental noise this expression is modified to

$$\delta C_l = \sqrt{\frac{1}{(2l+1)f_{sky}}} (C_l + C_l^{noise}) \quad (2.10)$$

where  $C_l^{noise}$  is the noise power spectrum. Typically, the detector (instrumental) noise sets a natural angular cutoff due to the final beam size,

$$C_l^{noise} = (\Delta_T \theta_b)^2 e^{l(l+1)\sigma_b^2} \quad (2.11)$$

where  $\theta_b = \sqrt{8\ln(2)}\sigma_b$  is the beam full width at half maximum (FWHM), and  $\Delta_T$  characterizes the detector's white noise level in temperature units. Power at multipoles  $l > (\sigma_b)^{-1}$  is exponentially downweighted. In other words, the experiment is cosmic-variance-limited only for  $C_l \lesssim C_l^{noise}$ .

In this CMB case Eqs.(2.3)-(2.5) yield

$$\mathcal{L} = \exp \left[ -\frac{(\lambda - \lambda_0)^2}{2\sigma_\lambda^2} \right] \quad (2.12)$$

where

$$\sigma_\lambda = \left[ \sum_l f_{sky} \left( \frac{2l+1}{2} \right) \frac{(\partial C_l / \partial \lambda)^2}{(C_l^{tot})^2} \right]^{-1/2} \quad (2.13)$$

is the standard deviation of the parameter  $\lambda$ , i.e. its  $1\sigma$  statistical uncertainty, and for simplicity we define

$$C_l^{tot} \equiv C_l + C_l^{noise}. \quad (2.14)$$

Now, the bias in the parameter  $\lambda$  in the presence of unaccounted-for contribution to the observed power spectrum, i.e.  $C_l^{tot} \rightarrow C_l^{tot} + \Delta C_l$ , is easily derived from Eq. (2.7). We attempt at fitting the observed power spectra with the ‘wrong’ theoretical model. For example, we fit the primordial CMB ‘contaminated’ by cluster- or filament-induced temperature anisotropy, the thermal Sunyaev-Zeldovich (SZ) effect, with theoretical model that accounts for the CMB only, without inclusion of filament-induced power. Doing so will clearly result in shifting (biasing) the best-fit  $\lambda$ , i.e.  $\lambda_0 \rightarrow \lambda_0 + \delta_\lambda$ . Of interest is the dimensionless bias, i.e. the bias in units of nominal statistical uncertainty, which gauges the bias importance;  $\frac{\delta_\lambda}{\sigma_\lambda} > 1$  indicates a relatively large bias, whereas the bias is small if  $\frac{\delta_\lambda}{\sigma_\lambda} < 1$ . Following a similar procedure to Eqs.(2.6)-(2.8), when there is an unaccounted contribution to the power spectrum, the likelihood function is

$$\mathcal{L} = \exp \left[ -\sum_l \frac{[\Delta C_l + (\frac{\partial C_l}{\partial \lambda})(\lambda - \lambda_0)^2]^2}{2(\delta C_l)^2} \right] \quad (2.15)$$

and the bias in the parameter  $\lambda$  is

$$\lambda\sigma_\lambda^{-2} = \lambda_0\sigma_\lambda^{-2} - f_{sky} \sum_l \left( \frac{2l+1}{2} \right) \frac{\frac{\partial C_l}{\partial \lambda} \Delta C_l}{(C_l^{tot})^2}. \quad (2.16)$$

Therefore, the dimensionless bias reads

$$\frac{\delta_\lambda}{\sigma_\lambda} = -f_{sky}\sigma_\lambda \sum_l \left( \frac{2l+1}{2} \right) \frac{\frac{\partial C_l}{\partial \lambda} \Delta C_l}{(C_l^{tot})^2}. \quad (2.17)$$

Biased cosmological parameter inference from CMB probes have already been discussed in the context of patchy reionization models (e.g. [6-7]) and residual ‘contamination’ of the CMB sky by undetected galaxy clusters [8-9].

It is constructive at this point to consider a toy model that will provide an order of magnitude estimate of how large can the bias be. Assume for this toy model only that both  $C_l$ ,  $\frac{\partial C_l}{\partial \lambda}$  and  $\Delta C_l$  are independent of  $l$ . In this case, Eqs.(2.13) & (2.17) give

$$\begin{aligned} \sigma_\lambda &= \left( \frac{f_{sky}}{2} l_{max}^2 \right)^{-1/2} \left( \frac{\partial C / \partial \lambda}{C} \right)^{-1} \\ \frac{\delta_\lambda}{\sigma_\lambda} &= \sqrt{\frac{f_{sky}}{2}} l_m \frac{\Delta C}{C} \end{aligned} \quad (2.18)$$

where here  $l_m$  is the effective maximum  $l$ .

For PLANCK and a CVL experiment,  $l_m \approx 2500 - 3000$  and  $f_{sky} \sim 1$ . This implies that for a fixed  $\delta_\lambda / \sigma_\lambda \approx 1$  the requirement is that  $|\Delta C|/C \lesssim 1/l$ , and that even if  $|\Delta C|/C$  is at the 0.1% level, we expect the dimensionless bias to be of order unity, i.e. it starts competing with the statistical uncertainty. Clearly, all power spectra in our case are  $l$ -dependent and this toy model does not apply. Nevertheless, it illustrates that while the large number of  $l$ -modes decreases the statistical uncertainty (as  $l_m^{-1}$ ), the bias is hardly affected. As a result, the dimensionless bias scales as  $l_m$ . We note that CMB power spectra have recently been sensitively measured at multipoles up to  $l = 10^4$  with the South Pole Telescope (SPT) and Atacama Cosmology Telescope (ACT) high-resolution ground-based telescopes.

The impact of bias induced by uncertainties in modeling the evolution of the LSS and its properties is determined from analysis of the matter power spectrum,  $P(k)$ . From Eq.(2.4) the Fisher matrix is

$$F_{ij} = \int_{k_{min}}^{k_{max}} \frac{\partial \ln(P)}{\partial \lambda_i} \frac{\partial \ln(P)}{\partial \lambda_j} dN_k \quad (2.19)$$

where [10]

$$\begin{aligned} dN_k &= V_{eff}(k) \frac{2k^2 dk}{(2\pi)^2} \\ V_{eff}(k) &= \int_{z=z_{min}}^{z_{max}} \left( \frac{n(z)P(k, z)}{1 + n(z)P(k, z)} \right)^2 \frac{dV}{dz} dz, \end{aligned} \quad (2.20)$$

and  $k_{max}$  and  $k_{min}$  mark the smallest and the largest scale in the survey. The number of modes sampled by the probe is set by  $k_{max}$ , and is roughly  $\sim \frac{4}{3} k_{max}^3 \frac{V_{eff}}{(2\pi)^3}$ . The analog of

Eq.(2.18) in this case is

$$\begin{aligned}\sigma_\lambda &\sim \left[ N \left( \frac{\partial P / \partial \lambda}{P} \right)^2 \right]^{-\frac{1}{2}} \\ \frac{\delta_\lambda}{\sigma_\lambda} &\sim N^{1/2} \frac{\Delta P}{P}.\end{aligned}\tag{2.21}$$

Combining the results from Eqs.(2.18) & (2.21) we arrive at the basic requirement that the fractional error in either the angular or matter power spectrum should satisfy

$$\begin{aligned}\frac{|\Delta C_l|}{C_l} &\lesssim \frac{1}{\sqrt{N}} \\ \frac{|\Delta P(k)|}{P(k)} &\lesssim \frac{1}{\sqrt{N}}\end{aligned}\tag{2.22}$$

for each  $l$  or  $k$ , namely that the fractional model error in the power spectra should be smaller than the square root of the number of modes. (Note that the total number of modes in a CMB experiment scales as  $\sim l_{max}^2$ , e.g. Eq. 2.13.)

It is a standard practice in the literature to show that the power spectra of systematics, foreground residuals, modeling errors, etc., are suppressed to below the cosmic variance level. This is warranted by a marginalization procedure over the systematics that results in what is presumed to be an unbiased estimate of the cosmological parameters for a relatively low cost of (usually) insignificant increase in the statistical uncertainty of the inferred cosmological parameters. However, in doing so it is tacitly assumed that the systematics model (or the residual systematics model) statistically fluctuates around the exact model; this assumption is rarely the case: Significant variation between theoretical models for statistical measures of the SZ effect is a relevant illustrative example. Predictions from these models are never found to fluctuate around each other. Rather, for virtually any two models for the SZ power spectrum (normalized to the same level at a given scale, e.g.  $l = 3000$ ) one typically finds that one of the models overestimates the power on small scales while the other overestimates it on larger scales. In other words, two different SZ models will typically have a different shape in multipole space, effectively undermining the basic assumption behind the marginalization procedure. If this marginalization path is nevertheless adopted, it would lead to an unrealistic level of uncertainty (which is derived in Appendix A),

$$\begin{aligned}\frac{\delta C_l^{sys}}{C_l} &\lesssim N^{-1/4} \\ \frac{\delta P(k)^{sys}}{P(k)} &\lesssim N^{-1/3},\end{aligned}\tag{2.23}$$

bounds that are weaker than the bias-free parameter inference requirements, Eq.(2.22). The reason for this is that in deriving Eqs.(2.22) we consider only that part of systematics, foregrounds, or modeling errors, that *systematically* increases or decreases the total power spectrum, in contrast to Eq.(2.23) which is obtained by assuming that all these systematics contribute power which fluctuates around the exact power spectrum.

### 3 CMB and LSS Precision Requirements

The above general assessment of the bias has important implications for the realistic degree of precision that can be attained in CMB (2D) and LSS (3D) probes, as we now demonstrate.



We consider the CMB as a representative for probes that are based on the angular power spectrum. Similar probes that will not be discussed here are weak gravitational lensing, and redshifted 21-cm analyses based on angular power spectra.

### 3.1 CMB Probes

As previously discussed by Seljak et al. [1], the required numerical precision of CMB Boltzmann codes at a given mode  $l$  is  $1/\sqrt{l}$  if the numerical errors are uncorrelated, i.e. fluctuating in  $l$ . If, on the other hand, there is a systematic (i.e.  $l$ -correlated) error in the power spectrum calculation it must satisfy

$$\frac{|\Delta C_l|}{C_l} \lesssim \frac{1}{l} \quad (3.1)$$

in order not to bias the parameter inference beyond the statistical error, at that given  $l$ . The argument is simply that of mode counting; in a mode-annulus of modulus  $l$  and width  $\Delta l$  there are  $2\pi l \Delta l$  modes (assuming statistical isotropy). The statistical error in estimating the angular power spectrum is therefore  $\delta C_l / C_l \sim 1/\sqrt{l}$  (assuming no mode-correlation). However, assuming  $\delta C_l$  are correlated, i.e. they are systematically lower or higher than the real power spectra, the requirement becomes  $|\Delta C_l| / C_l \lesssim 1/l$ . This will offset the  $\delta_\lambda / \sigma_\lambda \propto N^{1/2}$  dependence of the bias, where for the CMB - and other angular power spectra, such as those used in weak lensing shear maps or 21-cm forecast - the number of modes is  $N \sim l^2$ .

The primordial CMB power spectrum dies off very quickly beyond  $l \approx 1000$  and assuming a multipole cutoff  $l_{max} = 3000$  is reasonable. Comparing three Boltzmann codes, Seljak et al. [1] find that the 0.1% numerical precision is marginally achieved. Lesgourgues [11] illustrated that the CLASS and CAMB codes agree at the 0.01% level assuming the same evolutionary history. Recently, recombination modules for the CMB Boltzmann codes have been updated to include small corrections that resulted in only  $\sim 0.1\% - 0.2\%$  departures between CosmoRec [12-13] and HyRec [14-15], again marginally satisfying the bound, Eq.(2.22).

Achieving the goal of percent-level precision in determining the cosmological parameters may not be realistic given the various sources of systematics. It has recently been shown in [16] that the beam window function of PLANCK could be calibrated at the  $\sim 0.1\%$  level using a parametric beam model, but this degrades by a factor of a few if a non-parametric model is assumed. A systematic error higher than this benchmark will necessarily propagate into the recovered power spectra and will ultimately bias the inferred values of cosmological parameters. In Appendix B we provide a more quantitative discussion of the precision level that beam calibration has to satisfy in order not to violate Eq.(2.22).

The multifrequency capability of many CMB experiments will enable relatively precise removal of most astrophysical foregrounds, due to their non blackbody spectrum. A notable exception is the kinematic Sunyaev-Zeldovich (KSZ) effect which is essentially a first order Doppler shift of the CMB temperature, and therefore does not alter the blackbody spectrum. The estimated level of the KSZ from patchy reionization models is  $1.5 - 3.5 \mu K^2$  [17]. This range reflects the theoretical uncertainty in reionization models; it was obtained from studying the impact of  $\sim 100$  models on the CMB temperature anisotropy. The impact of astrophysical processes, degree of patchiness over the relevant redshift range, and various feedback processes, is estimated at the  $\sim 2 - 3 \mu K^2$  level at  $l = 3000$ , e.g. [18-21]. Overall, this range of variation of the KSZ power due to modeling uncertainty represents  $0.1 - 10\%$  perturbation to the primordial CMB on the relevant multipole range  $1000 < l < 3000$ . Using



the non-gaussianity of the SZ effect to remove this contribution is a reasonable possibility, but we are not aware of any quantitative study that demonstrates that the residual contribution to the CMB power spectrum will satisfy Eq.(2.22).

Taburet et al. [8] have shown that the thermal SZ effect, induced by hot gas in galaxy clusters, will significantly bias a few key cosmological parameters, even if the most luminous galaxy clusters detected by PLANCK are masked. This is due to the rather significant contribution made by the undetected clusters to the CMB temperature power spectrum and the finite number of frequency bands, instrumental noise, and foregrounds that limit the mass and redshift of detectable clusters. While this might not necessarily pose a significant challenge to PLANCK science, because cluster masking can benefit from non-CMB surveys that are projected to detect  $\sim O(10^5)$  clusters, it has recently been shown that warm filamentary structures may introduce comparable bias in the inferred cosmological parameters [9].

More generally, it remains to be demonstrated that the precision of foreground removal techniques can realistically attain the required level implied by Eq.(2.22). In the specific case of the thermal SZ effect in clusters, it should be emphasized that calculations of the power spectra do not agree at even the few percent level due to the highly model-dependent nature of the effect [22-24]. This applies not only to the amplitude but, more importantly, to the  $l$ -dependence of the SZ power spectrum. For example, Millea et al. [25] considered the possibility of mitigating the foregrounds-induced bias of inferred cosmological parameters by representing the foregrounds by 17 parameters which resulted in only  $\lesssim 20\%$  increase in the nominal uncertainty. While this result is encouraging, in order to assess the reliability of the inferred cosmological parameters from PLANCK, compelling evidence has to be provided that parametrizing the uncertainty in astrophysical foregrounds with only 17 parameters *fully* captures the  $l$ -dependence at the required  $|\Delta C_l|/C_l \lesssim 1/l$  level. As we show in Appendix C, allowing nuisance free parameters in a ‘wrong’ model does not generally guarantee a bias-free cosmological parameter inference.

### 3.2 LSS Surveys

3D surveys that target the matter power spectrum,  $P(k)$ , clearly probe a larger number of modes than the 2D CMB angular power spectrum. The number of modes is  $\sim \frac{4}{3}k_{max}^3 \frac{V_{eff}}{(2\pi)^3}$ , where (as specified in the previous section)  $V_{eff}$  and  $k_{max}$  are the effective survey volume and maximum wave number, respectively (Eqs. 2.20). For a  $\sim 1 Gpc^3$  survey, and  $k_{max} \sim 0.1 Mpc^{-1}$  the total number of modes is  $N \sim 5300$ , and the required precision on the matter power spectrum is  $|\Delta P(k)|/P(k) \lesssim 1.3\%$ . The SDSS Lumonius Red Galaxy (LRG) survey used 42,000 modes [26] in the  $0.01h/Mpc < k < 0.1h/Mpc$  range. Reid et al. [27] probed deeper into the quasi-linear regime and used data up to  $k < 0.2h/Mpc$ , which resulted in increasing the mode number by a factor of  $\sim 8$ , thereby significantly improving the constraints on the cosmological parameters. All these galaxies are observed at  $z < 1$ ; how well do we know the matter power spectrum at the quasi-linear regime? Calculation of the evolution of  $P(k; z)$  can be done perturbatively. Two-loop corrections still contribute at the  $\sim 1\%$  level at quasi-linear scales, e.g. [28]. Going to higher order in regularized perturbation theory entails performing integrations at  $D = 3n - 1$  dimensions where  $n$  is the loop order, e.g. [29]. This is expected to be computationally quite prohibitive if the desired numerical accuracy, Eq.(2.22), is to be achieved. For example, on scales where 3rd order perturbation terms are non-negligible, 8D integrations should be done that have to be precise at the  $10^{-3} - 10^{-7}$  level, say, and this will have to be repeated for each parameter set in the multi-dimensional

parameter-space search for the best-fit cosmological model. It is unclear whether this is realistic.

An alternative is to employ numerical simulations; carefully choosing initial conditions, time steps, sufficient mass resolution and large volumes will allow reaching the rather impressive 1% accuracy at  $k \sim 1 Mpc$  [30], but this without including the effect of baryons which in itself is expected to contribute at the percent level on these small scales. Also, to be useful for Monte-Carlo cosmological parameter search, these calculations of the nonlinear matter power spectrum have to be very fast. Running the simulations for only a few cases and interpolating between parameter values might result in errors larger than those allowed for unbiased cosmological parameters. In addition, a comparison of these simulations with CAMB’s HALOFIT reveals a 5-10% discrepancy [30].

Another possible avenue is to employ Artificial Neural Networks (ANN) for a fast calculation of the matter power spectrum. Agarwal et al. [31] claim to have reached the  $\lesssim 1\%$  precision on scales  $k \leq 0.7 h Mpc^{-1}$  and at redshifts  $z < 2$ . However, these neural networks have been trained with HALOFIT, which is itself discrepant with [30] at the few percent level.

In parameter estimation forecasts it is customary to adopt the nonlinear-scale cutoff at scales where the mass fluctuation inside mass spheres of radius  $R$ , is of order unity:  $\sigma(R) \equiv \sqrt{\langle (\frac{\delta M}{M})^2 \rangle_R} \approx 0.5$ , e.g. [32-33]. While this definition of nonlinear scale is intuitive, parameter bias is completely unaccounted for. In fact, it turns out to overly under-estimate the impact of our ignorance of the nonlinear matter power spectrum on the bias of inferred cosmological parameters. In their parameter estimation forecast for *post-reionization* redshifted-21-cm surveys, Visbal, Loeb, & Wyithe [34], determined the largest mode  $k_{max}$  by comparing the nonlinear matter power spectrum from HALOFIT to the linear power spectrum at the various redshifts they considered. They defined  $k_{max}$  at the scale where the nonlinear deviates from the linear power spectrum at 10%. They also explored the robustness of their forecast to varying this criterion in the range 5-25% and indeed their analysis shows that the parameter uncertainties degrade as  $k_{max}^{-3/2}$  (in the range where cosmic variance dominates over instrumental noise). Since their 21-cm observation is volume-limited, the number of modes can be readily calculated,  $N \sim 6.75 \times 10^{11} k_{max}^3$ , where  $k_{max}$  is in  $Mpc^{-1}$  units. Even if we take their most stringent  $k_{max} = 0.1 Mpc^{-1}$ , we obtain that for Eq.(2.22) to be satisfied one must know the matter power spectrum to better than one part in  $10^4$ . This implies that realizing the potential of future post-reionization redshifted-21-cm observations seems unlikely. Given the current level of precision in calculations of the matter power spectrum, the number of modes will have to be drastically reduced; this will result in a significant weakening the stated scientific yield of these probes, removing their competitive advantage over other cosmological probes.

The tantalizing merits of the pre-reionization 21-cm at very high redshifts, and its statistical power to constraining cosmology, have been first advocated by Loeb & Zaldarriaga [35]. One may contemplate that *pre-reionization* 21-cm at high-redshifts is immune to power spectrum non-linearity, which is indeed the case above some redshift-dependent sub-Mpc scale. However, use of data down to the baryon Jeans scales comes with the penalty of incurring a very large bias; the number of modes in these observations is estimated to fall in the range  $10^{14} - 10^{16}$ , which will require knowing the matter power spectrum at the  $\sim 10^{-7} - 10^{-8}$  precision. Indeed, it has been shown in [36] that nonlinear corrections to the matter power spectrum, even at redshifts as high as 30 or 50, may contribute at the

sub-percent level at sub-Mpc scales. As discussed above, while accounting for higher order corrections to the matter power spectrum is theoretically possible (even if at the cost of significantly slowing down the search in the multidimensional parameter space), it is not clear how many such terms should be included in order to reach the fantastic  $\sim 10^{-7} - 10^{-8}$  precision entailed by requiring unbiased parameter estimation from all scales down to the baryon Jeans scale. Cutting off the data above some scale  $k_{max}$  larger than the Jeans scale  $k_J$  will degrade the statistical uncertainty by  $(\frac{k_{max}}{k_J})^{3/2}$ .

These considerations are especially relevant when assessing the scientific yields of future surveys. For example, Sigurdson & Cooray [37] considered the possibility of delensing the polarized CMB sky with high-redshifted 21-cm tracer of gravitational lenses to the level that will allow constraining the energy scale of inflation down to  $\mathcal{V}^{1/4} \sim 1.1 \times 10^{15}$  GeV, equivalent to tensor-to-scalar ratio  $\mathcal{T}/\mathcal{S} \sim 1.0 \times 10^{-6}$  with  $l_{max} \sim 10^5$ . This represents  $\sim 3$  orders of magnitude tighter constraint than the ideal CMB experiment. However, reliably inferring the energy scale of inflation using this method requires a  $\sim 10^{-5}$  precision of the theoretical 21-cm anisotropy model, a goal that we believe has not been demonstrated to be realistic. In a recent similar work, using  $l_{max} \sim 10^7$ , Book, Kamionkowski & Schmidt [38] claim that the signature of inflationary primordial gravitational waves with as small as  $\mathcal{T}/\mathcal{S} \sim 1.0 \times 10^{-9}$  could be detected with future high-redshifted 21-cm observations. This further boosts the model precision requirement to one part in  $10^7$ .

It is well appreciated that realizing the potential of the 21-cm probe will be extremely challenging given that foregrounds are expected to be  $\sim 5$  orders of magnitude larger than the 21-cm signal. In Fisher matrix forecasts of the science yield of these 21-cm observations, it is common to model the frequency-dependence and spatial correlations of these foregrounds and marginalize over the model free parameters, e.g. [2], [4], [39-42]. These models are often extrapolated from other frequency regimes, or are otherwise only partially physically motivated, and are often proposed largely due to their functional simplicity. However, for unbiased cosmological parameter estimation the marginalization process only makes sense when the model functional form, i.e.  $k$ -dependence, faithfully captures the shape of the observed power spectrum; the extra freedom enabled by adding the nuisance parameters, which are subject to marginalization, does not guarantee a bias-free parameter inference. In other words, we generally do not expect that using an inadequate model can be *fully* compensated for by simply marginalizing over free nuisance parameters (as argued in Appendix C).

So far we considered only the systematics in the primordial matter power spectrum. In reality, the matter power spectrum is determined from observations of dark matter biased tracers, e.g., galaxy clustering, galaxy clusters, and perhaps also from redshifted 21-cm neutral gas (that follows dark matter halos) in the future. The *observed* power spectrum is skewed by a scale-dependent multiplicative bias. The observable in galaxy surveys is  $P_g(k; z) = b^2(k, z)P(k; z)$ , e.g. [43]. The calculation of this luminosity-dependent bias is highly non-trivial and model-dependent, so one may question its precision. In the case of 21-cm observations at the reionization era, we assume the surveys to be only volume-limited; this eases the bias calculation in the sense that it is only redshift dependent in that case. For far-future high-redshift 21-cm observations at high redshifts, there is very little halo bias and in that sense this probe is more immune to this systematic source.

For the bias  $b(z)$  calculation one needs to specify the halo mass function. Until only recently the mass function of choice was that of Seth & Tormen [44-45]. This was superseded by the Tinker mass function [46-47], but there are several other mass functions. Typically, they are discrepant at more than a few percent on the relevant mass-redshift range. In most

forthcoming analyses we will only reliably know the redshifts of galaxies or galaxy clusters; their mass inference is expected to be very challenging and quite biased. When comparing observations to theory a mass function has to be selected in order to average the theoretical bias  $b(M, z)$  over mass. Also, it is not clear how this average should be performed; is it a simple mass average, e.g. [48] ? Is it mass-weighted average over redshift, e.g. [34] ? It is therefore clear that uncertainties in mass and in the mass function imply the need for arbitrary choices that may easily result in a few percent systematic bias in all  $k$ -modes, ultimately biasing the inferred cosmological parameters.

## 4 Discussion

The concordance cosmological model, corroborated by many different cosmological surveys that probe the linear and quasi-linear large scale structure, is surprisingly simple on the largest scales: Of all conceivable universes ours seems to have had initial conditions that lead to relatively simple LSS, and with properties that are characterized by only a dozen cosmological parameters. It now appears that our most important challenge is precision measurements of these parameters. The ability to do so depends largely on the sensitivity and extent of future cosmological surveys. More specifically, the level of precision depends on the susceptibility of the cosmological model to small variations in the key parameters, the number of independent (Fourier) modes of survey data, and on the degeneracy between the parameters.

In reality various factors can limit the quality of our deductions from even the most precise measurements; almost unavoidably these include some degree of bias in the inferred cosmological parameters. In this work we highlighted the quantitative ‘tension’ between the statistical error that drops as  $N^{-1/2}$  and the dimensionless bias that typically grows as  $N^{1/2}$ , where the number of modes,  $N$ , increases as cosmological experiments become significantly more sensitive. Realizing the potential of redshifted 21-cm experiments that will explore  $10^8 - 10^{14}$  modes requires controlling various systematics, including modeling and simulation uncertainties and foreground removal, to the level of  $O(10^{-4})$ - $O(10^{-7})$ . Put differently, whereas a model bias of  $0.001\sigma$  could be identified with the CMB satellite WMAP ( $\sim 10^6$  modes) at  $1\sigma$ , a bias as small as  $10^{-7}\sigma$  would be discerned with future 21 cm observations ( $\sim 10^{14}$  modes); a model bias larger than this would therefore be at odds with observation and in any case would bias the inferred cosmological parameters. As emphasized above, our main concern in this work has been those uncertainties that systematically shift the power spectra in an unknown fashion (for which we adopt Eq. (2.22) as the benchmark precision criterion). We stress that systematics that fluctuate around the exact model can be marginalized over and typically result in mild degradation of the nominal cosmological parameter uncertainties without inducing any bias [in which case the required precision is summarized in Eq. (2.23)]. However, for this to be the case the model has to exactly capture the shape of the power spectra over the *entire* range of variation of the cosmological parameters. Only then could the model nuisance parameters be marginalized over and ensure a bias-free cosmological parameter inference.

With the steadily increasing number of accessible CMB and LSS modes the statistical uncertainty in inferred cosmological parameters is ought to improve. However, even a slight bias in the theoretical modeling, data analysis, and foreground removal, can potentially accrue with increasing number of modes to the level that may possibly bias the best-fit cosmological model far beyond the nominal statistical uncertainty. This poses a challenge to

next generation LSS probes, especially to redshifted 21 cm observations which are theoretically limited only by the baryon Jeans scale, and are poised to ultimately yield measurement database covering a huge number of  $\sim 10^{16}$  modes with unprecedented capability for precise cosmological parameter inference.

The simple rule of thumb that we highlighted in this work, that the statistical error and dimensionless bias on the inferred cosmological parameters scale as  $\sim N^{-1/2}$  and  $\sim N^{1/2}$ , respectively, assumes that the cosmological information is entirely contained in the angular or matter power spectrum. However, constraining primordial non-gaussianity requires working with the angular (matter) bispectrum, in the 2D (3D) cases. When the main goal is just to set observational bound on the degree of non-gaussianity, without specifying the non-gaussianity class, then all triangular configurations in mode-space are allowed, thereby increasing the number of modes  $N$  (for a given observational resolution) used in determining the primordial non-gaussianity. This only makes the theoretical precision requirements from the model (which is contrasted with the data) more demanding.

Our objective in this work has been to highlight a major limitation of upcoming cosmological surveys, especially those that are expected to greatly benefit from the huge number of modes hitherto unprobed by the current lower resolution and higher noise probes. Strictly speaking, our simple arguments apply to either a cosmological model with a single free parameter, or to multi-parameter model when the parameters are uncorrelated. In practice, this is never the case and cosmological parameters do correlate. As the number of usable modes increases there is more information in the data to allow breaking the cosmological parameter degeneracies. However, as we argue in this work, using a larger number of modes could lead to a stronger bias if the model is not sufficiently accurate. To optimize the number of modes used in a given cosmological survey the maximal number of modes should be chosen such that the statistical error and bias added in quadrature will result in the minimum possible error on a given set of parameters.

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## A Precision Requirements: Random Modeling Errors

We outline the rationale behind the standard requirement on precision of theoretical power spectra, pointing out explicitly the assumption behind the derivation of Eq.(2.23). We do this for the case of angular power spectrum; extending this derivation to the case of the matter power spectrum is straightforward. Analogous to parameter marginalization one can account for residual foreground (or other type of modeling uncertainty) by simply modifying Eq.(2.15)

$$\mathcal{L}' = \int_{\Delta C_l} \exp \left[ - \sum_l \frac{[\Delta C_l + (\frac{\partial C_l}{\partial \lambda})(\lambda - \lambda_0)^2]^2}{2(\delta C_l)^2} \right] \exp \left( - \frac{\Delta C_l^2}{2\sigma_{C_l}^2} \right) \mathcal{D}(\Delta C_l), \quad (\text{A.1})$$

where we marginalize over the  $l$ -dependent modeling error of the power spectrum (assuming that foregrounds and other systematics have already been accounted for) and carry out a



functional integration over it with a gaussian prior on the power spectrum modeling uncertainty with  $\sigma_{C_l}$  being the  $1 - \sigma$  prior on the residual modeling error  $\Delta C_l$ . For simplicity we assume it is symmetrically distributed around 0, since otherwise this itself would introduce a bias. Carrying out the integration one obtains that  $\delta C_l^2$  in Eq.(2.15) is replaced by

$$\delta C_l^2 \rightarrow \delta C_l^2 + \sigma_{C_l}^2. \quad (\text{A.2})$$

Thus, the inferred parameter is unbiased at the cost of increasing variance. As  $\delta C_l^2$  decreases as  $l^{-1}$ , we expect the residual SZ power with  $\sigma_{C_l}^2$  to dominate the denominator (in the above sum) at some sufficiently large  $l$ , resulting in degraded statistical uncertainty of several cosmological parameters. This would typically be perceived as a reasonable cost when compared to the bias that would otherwise be introduced. The first relation in Eq. (2.23) can be obtained from the following consideration: If  $\delta C_l^2 \propto C_l^2/l$  then the requirement is  $\sigma_{C_l}/C_l \lesssim l^{-1/2}$ , and recalling that the number of modes  $N \sim l_{max}^2$  it follows that  $\delta C_l/C_l \lesssim N^{-1/4}$ . It is straightforward to obtain also the second relation in Eqs.(2.23) by employing similar reasoning for the 3D matter power spectrum,  $P(k)$ , and recalling that here  $N \sim k_{max}^3 V$ . For very large mode numbers,  $N$ , Eqs.(2.22) & (2.23) give markedly different requirements. In this Appendix we show that making the requirement Eq.(2.23) is only warranted under the very strong assumption that the residual systematic power spectrum fluctuates around 0. In general, our modeling uncertainty of systematics, foreground residual, etc., is hardly of this type; in fact, we rarely know the systematic power spectrum up to a fluctuating part. Relevant examples include the statistical thermal SZ effect, the KSZ effect from patchy reionization, foreground clustering, and beam systematics.

## B Beam Calibration Requirements

Features in sky maps on scales smaller than the angular resolution of the telescope are effectively smeared by beam dilution. Conventionally, the telescope angular response function is modeled as a circular or elliptical gaussian with superimposed low order polynomials. By calibrating the beam against a point source such as Mars, Jupiter, or Saturn, the best fit model parameters are obtained.

For simplicity, we model the beam dilution effect as a circular gaussian, but the result will hold for more general beam models. The *measured* power spectrum obtained from the map is

$$C_l^{measured} = C_l^{real} e^{-l^2 \sigma_b^2} \quad (\text{B.1})$$

where  $\sigma_b$  is the gaussian width. Once the beam is calibrated and  $\sigma_b$  is obtained the measured power spectrum is multiplied by  $e^{l^2 \sigma_b^2}$  and the best-fit cosmological model is then obtained from comparing theory and observation. In practice, the model chosen for the beam description might not always capture all beam features, such as sidelobes, higher order moments beyond the quadrupole (parameterized by the beam ellipticity), and non-gaussian features, etc. Choosing the ‘wrong’ model might then skew the best-fit procedure. It is conceivable that the beam calibration will then result in an effective beam width  $\sigma_{b'}$  either larger or smaller than  $\sigma_b$ . In this case, the recovered power spectrum will be

$$C_l^{recovered} = C_l^{measured} e^{l^2 \sigma_{b'}^2} \quad (\text{B.2})$$

which is systematically larger or smaller than the actual power spectrum  $C_l^{measured} e^{l^2 \sigma_b^2}$ . The fractional error in the angular power spectrum is therefore

$$\frac{\Delta C_l}{C_l} = \exp[l^2(\sigma_b^2 - \sigma_{b'}^2)] - 1. \quad (\text{B.3})$$

Assuming the difference  $\sigma_b - \sigma_{b'}$  is small, and combining this with Eq.(2.22) results in the condition

$$2l^2 \sigma_b^2 \mu \lesssim 1/l \quad (\text{B.4})$$

where we defined the beamwidth mismatch  $\mu = |\sigma_b - \sigma_{b'}|/\sigma_b$ . Now, recalling that the maximum multipole probed by the telescope is roughly where the beam window function drops to a value  $e^{-1}$  times its peak value, we can set  $l_{max}^2 \sigma_b^2 \approx 1$  in Eq.(B.4) and obtain

$$2\mu \lesssim 1/l_{max} \quad (\text{B.5})$$

implying that for a beamsize of  $\sim 5$  arcminute (the smallest of the PLANCK/HFI instruments) the beamwidth has to be calibrated at the 0.1 arcsecond precision, which will be challenging given that point sources (such as planets) morphology in the microwave is not known to this very high precision level.

## C Marginalization and Residual Bias

Marginalizing over model uncertainties is a standard practice in analyses/forecasts of cosmological datasets. This technique is especially relevant for addressing *residual* foregrounds or systematics. Accounting for these model uncertainties is conventionally done by parameterizing systematics and foreground models. In most cases these somewhat *arbitrary* model choices are motivated by either mathematical simplicity or well-understood physics *extrapolated* to the regime of interest. As mentioned in Section 3 (in our discussion of the CMB and 21-cm surveys), this procedure may not meet the precision standards required by unbiased cosmological parameter inference, as put forward in this work.

Here we argue that the bias derived in section 2 cannot be simply integrated away by *assuming* a model for the foregrounds and systematics with free nuisance parameters. Therefore, although Fisher matrix analyses of the future performance of marginalization techniques applied to CMB and 21-cm observations typically result in only a mild increase in the statistical error, this procedure by no means alleviates the bias problem.

Generalizing Eq.(2.6), we denote the model  $\Delta d_{mod}$  for the systematic  $\Delta d$ , so that the likelihood function can now be written as

$$\mathcal{L} \rightarrow \mathcal{L}' = \exp \left[ -\frac{[\Delta d - \Delta d_{mod} + \frac{d(d)}{d\lambda} \cdot (\lambda - \lambda_0)]^2}{2(\delta d)^2} \right]. \quad (\text{C.1})$$

We further parameterize  $\Delta d_{mod} = A \Delta \tilde{d}$  with  $A$  denoting a nuisance model parameter that we marginalize over, assuming a gaussian prior, with the dependence on the Fourier mode ( $l$  or  $k$ ) included in  $\Delta \tilde{d}$

$$\mathcal{L}' \rightarrow \mathcal{L}'' = \int dA \exp \left[ -\frac{[\Delta d - A \Delta \tilde{d} + \frac{d(d)}{d\lambda} \cdot (\lambda - \lambda_0)]^2}{2(\delta d)^2} - \frac{(A - A_0)^2}{2\sigma_A^2} \right] \quad (\text{C.2})$$



where  $A_0$  and  $\sigma_A$  characterize the prior range for the nuisance parameter  $A$ . This parameter can, for example, be  $A_{SZ}$  of the recent SPT and ACT parameterization of the amplitude of the SZ angular power spectrum. In this case it is unlikely that N-body simulations or analytic modeling will result in an SZ angular power spectrum shape  $\Delta\tilde{d}$  identical to the actual  $\Delta d$  to the required precision  $|\Delta C_l^{SZ}|/C_l \lesssim 1/l$  up to the maximum  $l \sim 3000$  or  $4000$  used in the SZ analysis. Carrying out the integration over  $A$  in Eq.(C.2) in the range  $[-\infty, \infty]$  is partially justified by assuming that it is known to be several  $\sigma$  above zero, and partially warranted by the fact that if we set the lower integration limit to zero (or a sufficiently small value) then the integration over  $A$  in an asymmetric range will only weaken the ‘power’ of marginalization, resulting in even a larger bias than what we estimate here. We obtain

$$\mathcal{L}'' = \exp \left[ -\frac{\beta^2}{2} \frac{1}{(\delta d)^2 + \sigma_\alpha^2 (\Delta\tilde{d})^2} \right] \quad (\text{C.3})$$

where

$$\beta = \Delta d - A_0 \Delta\tilde{d} + \frac{d(d)}{d\lambda} \cdot (\lambda - \lambda_0). \quad (\text{C.4})$$

Comparing this to Eqs.(2.6) we see that the bias and statistical uncertainty changed

$$\begin{aligned} \Delta d &\rightarrow \Delta d - A_0 \Delta\tilde{d} \\ (\delta d)^2 &\rightarrow (\delta d)^2 + \sigma_\alpha^2 (\Delta\tilde{d})^2. \end{aligned} \quad (\text{C.5})$$

Typically,  $\delta d$  increases by a factor of order unity, e.g. [4], [25], and while one wants to use a model where  $|\Delta d - A_0 \Delta\tilde{d}| \lesssim 1/\sqrt{N}$ , this is seldom the case (if at all). In the case of CMB and its SZ foreground, different SZ models do not even agree (in amplitude and shape) to better than a few tens of percent, far above the  $|\Delta d - A_0 \Delta\tilde{d}| \lesssim 1/l_{max}$  requirement.

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